Deformation of 2-Step Nilmanifolds with Abelian Complex Structures

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Abstract. We develop deformation theory for abelian invariant complex structures on a nilmanifold, and prove that in this case the invariance property is preserved by the Kuranishi process. A purely algebraic condition characterizes the deformations leading again to abelian structures, and we prove that such deformations are unobstructed. Various examples illustrate the resulting theory, and the behavior possible in 3 complex dimensions.

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1 Introduction

In this paper, we study complex structures associated to compact quotients of nilpotent groups. These manifolds are called nilmanifolds, and an investigation of the special class of Kodaira manifolds was completed in [7]. The present paper opens this discussion to a wider class of nilmanifolds.

A left-invariant complex structure on a Lie group is said to be *abelian* if the complex space of (1,0)-vectors is an abelian algebra with respect to Lie bracket. This definition only makes sense in the algebraic setting, and in this context it is of particular interest to know to what extent a study of *invariant* complex structures on nilmanifolds captures the general situation.

There is a total of six 6-dimensional 2-step nilpotent groups admitting abelian complex structures [14]. If R^n denotes a n-dimensional abelian group and H_{2n+1} a $(2n+1)$-dimensional Heisenberg group, then the 2-step nilpotent groups with abelian complex structures are R^6, $H_5 \times R^1$, $H_3 \times R^3$, $H_3 \times H_3$, the Iwasawa group W_6 and one additional group which we denote by P_6. These groups are the 6-dimensional instances of the respective series: R^{2n}, $H_{2n+1} \times R^{2m-1}$, $H_{2n+1} \times H_{2m+1}$, W_{4N+2} and P_{4N+2}. For example, W_{4N+2} is the real group underlying a generalized complex Heisenberg group.

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The compact quotients of R^{2n} are complex tori, and their deformation and moduli are well studied [3]. A detailed account of the moduli space of complex structures of a special compact quotient of $H_{2n+1} \times R$ was recently given in [7], and in [8] a somewhat different method is directed towards the study of W_6. The present paper helps to unite these two approaches, particularly via the examples in §6.

This paper presents a general approach to computing the deformations of 2-step nilmanifolds with abelian complex structures. To analyze data on deformation theory, our first step is to identify the Dolbeault cohomology of a 2-step nilmanifold with abelian complex structure with the appropriate Lie algebra cohomology (Theorem 1). General results of this nature were proved in [4], though we shall need our own explicit description of this identification. The second step is to extend this identification to the determination of harmonic representatives for Dolbeault cohomology with coefficients in the tangent sheaf (Theorem 3).

We derive the first main result of this paper in §4.3:

Theorem *Let G be a 2-step nilpotent Lie group with co-compact subgroup Γ. Then any abelian invariant complex structure on $X = \Gamma \backslash G$ has a locally complete family of deformations consisting entirely of invariant complex structures.*

This is proved by showing that an application of Kuranishi's method does not take one outside the subspace of invariant tensors. The theorem implies that any deformation of an abelian invariant complex structure is necessarily equivalent to an invariant one, at least of the deformation is sufficiently small.

Given this result, it makes sense to ask under what conditions the deformed invariant structures remain abelian. Indeed, our techniques enable us to prove that deformations preserving the abelian property are always unobstructed and faithfully represented at the infinitesimal level:

Theorem *On a 2-step nilmanifold X with an abelian complex structure, a vector in the virtual parameter space $H^1(X, \Theta_X)$ is integrable to a 1-parameter family of abelian complex structures if and only if it lies in a linear subspace defining the abelian condition infinitesimally.*

The 'only if' part is obvious, but the force of this result is the backwards implication. Once we convert to Lie algebra cohomology, it reduces the constraints of abelian deformations to purely algebraic equations that we introduce in §5 and collectively call 'Condition A'. Using this, one may carry out an effective computation in terms of structural constants of the nilpotent groups in question.

Further analysis of abelian deformations yields a characterization of the Kodaira manifolds (defined in §2.1) as those corresponding to a Lie algebra with 1-dimensional center and for which all infinitesimal parameters are integrable and lead to abelian deformations. The precise statement is Theorem 6 in §5.1.

In the final section, we compute the relevant cohomology dimensions for a number of nil 6-manifolds, each equipped with a natural abelian complex structure. In so doing, we are able to compare the techniques of this article with those of [14], but we emphasize a complication that arises from a choice of complex structure with added symmetry. The first theorem above allows one to dispense with the Kuranishi method in the explicit construction of parameter spaces, and replace it with a more direct calculation involving invariant differential forms. This we do in Example 8, after having first illustrated the power of the second theorem above in estimating the dimension of a potential moduli space.

2 Abelian complex structures

Suppose that a Lie algebra $\mathfrak{g}$ admits an endomorphism J such that

$$J \circ J = -1 \quad \text{and} \quad [JA, JB] = [A, B] \tag{1}$$

for all A, B in $\mathfrak{g}$. It can be extended by left-translation to an endomorphism of the entire tangent bundle of G. Then J defines an invariant almost complex structure on the group G which is integrable, since (1) implies the vanishing of the Nijenhuis tensor. A complex structure satisfying (1) is called *abelian*, and the identity also implies that the center $\mathfrak{c}$ is J-invariant.

Now assume that the Lie algebra is 2-step nilpotent. In particular, the first derived algebra is contained in the center. Taking the quotient of the algebra $\mathfrak{g}$ with respect to the center, we obtain an abelian algebra $\mathfrak{t}$. When the complex structure is abelian, it induces a complex structure on $\mathfrak{t}$. The identities

$$\mathfrak{g}^{(1,0)} = \mathfrak{t}^{(1,0)} \oplus \mathfrak{c}^{(1,0)} \quad \text{and} \quad \mathfrak{g}^{(0,1)} = \mathfrak{t}^{(0,1)} \oplus \mathfrak{c}^{(0,1)}$$

concerning type (1,0) and (0,1) vectors are therefore valid at the level of vector spaces.

Let $\{X_i, JX_i : 1 \leqslant i \leqslant n\}$ be a real basis for $\mathfrak{t}$ and $\{Z_\alpha, JZ_\alpha : n+1 \leqslant \alpha \leqslant n+m\}$ a real basis for $\mathfrak{c}$. A basis of $(1,0)$ vectors for the complex tangent bundle of G is composed of the elements

$$T_j = \tfrac{1}{2}(X_j - iJX_j) \quad \text{and} \quad W_\alpha = \tfrac{1}{2}(Z_\alpha - iJZ_\alpha). \tag{2}$$

The complex structural constants E^α_{kj} and F^α_{kj} are defined by

$$[\overline{T}_k, T_j] = \sum_\alpha E^\alpha_{kj} W_\alpha + \sum_\alpha F^\alpha_{kj} \overline{W}_\alpha, \tag{3}$$

and satisfy

$$F^\alpha_{kj} = -\overline{E}^\alpha_{jk}.$$

We continue to use Roman indices in the range $1, \ldots, n$ and Greek indices for $n+1, \ldots, n+m$. Let ω^j be the left-invariant (1,0)-forms dual to the vectors T_j for $1 \leqslant j \leqslant n$ and annihilating the W_α. They span the space $\mathfrak{t}^{*(1,0)}$. Similarly, there are left-invariant (1,0)-forms ω^α dual to W_α, annihilating the T_j. The dual form of the structural equation (3) is

$$d\omega^\alpha = \sum_{i,j} E^\alpha_{ji} \omega^i \wedge \overline{\omega}^j. \tag{4}$$

The forms ω^i are all exact, and $\partial\omega^\alpha = 0$. Thus,

Lemma 1. *The forms $\overline{\omega}^1, \ldots, \overline{\omega}^n, \overline{\omega}^{n+1}, \ldots, \overline{\omega}^{n+m}$ are all $\overline{\partial}$-closed.*

Now suppose that there exists a discrete subgroup Γ of G such that the left quotient space $\Gamma\backslash G$ is compact. The resulting quotient is called a *nilmanifold.* Since the complex structure J is left-invariant, it descends to a complex structure on $X = \Gamma\backslash G$. Such a discrete subgroup always exists if there is a basis such that the real structural constants are rational [11].

Later in this paper, we shall study the deformation theory on such compact complex manifolds. At an appropriate juncture (in §4.1), we shall find it convenient to introduce an invariant Hermitian metric on X. We shall choose such a metric so that $\{X_j, JX_j, Z_\alpha, JZ_\alpha\}$ forms a Hermitian frame. First we describe some simple examples of nilmanifolds and complex structures.

2.1 Kodaira manifolds and other examples

Our first example of a compact nilmanifold with an abelian complex structure is a Kodaira manifold, a generalization of a Kodaira surface. We proceed to list algebraic constructions of this and similar examples.

Example 1. On the vector space $\mathbb{R}^{2n+2}$ with basis $\{X_j, Y_j, Z, A\}$, define a Lie algebra by setting

$$[X_j, Y_j] = -[Y_j, X_j] = Z, \qquad 1 \leqslant j \leqslant n,$$

and declaring all other brackets to be zero. This turns the vector space into the direct sum $\mathfrak{g} = \mathfrak{h}_{2n+1} \oplus \mathfrak{t}_1$ of the Heisenberg algebra and the 1-dimensional algebra.

We define an almost complex structure J on the Lie algebra $\mathfrak{g}$ by means of the equations

$$JX_j = Y_j, \quad JZ = A, \tag{5}$$

so that the equations $JY_j = -X_j$ and $JA = -Z$ are also understood. The endomorphism J defines an abelian complex structure on $\mathfrak{g}$, and therefore on the manifold $H_{2n+1} \times R$ and a compact quotient thereof. In the niotation (2), its complex structure equation is

$$[\overline{T}_j, T_j] = -\tfrac{1}{2} i (W + \overline{W}). \tag{6}$$

The moduli problem of the compact quotient of such complex manifolds was studied extensively in [7].

We next describe a more general extension of the Heisenberg group, and then a product example.

Example 2. Let $H_{2n+1} \times R^{2m+1}$ be the product of a real Heisenberg group and an abelian Lie group with dimensions as specified. Let $\{X_j, Y_j, Z\}$ be a basis for $\mathfrak{h}_{2n+1}$, and let $\{Z_0, Z_{2k-1}, Z_{2k}\}$ (with $1 \leqslant k \leqslant m$) be a basis for R^{2m+1}. The non-zero structural constants are determined by the single set of equations

$$[X_j, Y_j] = Z.$$

An abelian complex structure is defined by setting

$$JX_j = Y_j, \quad JZ = Z_0, \quad JZ_{2k-1} = Z_{2k}, \tag{7}$$

in analogy to (5).

Example 3. The product $H_{2n+1} \times H_{2m+1}$ is a 2-step nilpotent group with 2-dimensional center. Let $\{X_j, Y_j, Z_1, X_k, Y_k, Z_2\}$ (with $1 \leqslant j \leqslant n, 1 \leqslant k \leqslant m$) be a basis for its Lie algebra $\mathfrak{h}_{2n+1} \oplus \mathfrak{h}_{2m+1}$. The non-zero structural constants are

$$[X_j, Y_j] = Z_1 \quad \text{and} \quad [X_k, Y_k] = Z_2.$$

Define an almost complex structure J on this space by

$$JX_j = Y_j, \quad JX_k = Y_k, \quad JZ_1 = Z_2; \tag{8}$$

once again this defines an abelian complex structure.

We describe the remaining classes of examples in 6 dimensions for simplicity.

Example 4. The group structure of W_6 underlies that of the complex Heisenberg group. On the algebra level, the structural equations of W_6 are

$$[X_1, X_3] = -\tfrac{1}{2}Z_1, \quad [X_1, X_4] = -\tfrac{1}{2}Z_2, \quad [X_2, X_3] = -\tfrac{1}{2}Z_2, \quad [X_2, X_4] = \tfrac{1}{2}Z_1. \tag{9}$$

An abelian complex structure is defined by

$$JX_1 = X_2, \quad JX_3 = -X_4, \quad JZ_1 = -Z_2, \tag{10}$$

and this is denoted J_1 in [8]. Beware that J is not the standard bi-invariant complex structure J_0 that makes W_6 a complex Lie group; indeed

$$J_0\,[A, B] = [J_0 A, B]\,, \qquad A, B \in \mathfrak{g},$$

and so J_0 is definitely not abelian. Nonetheless, both J_0 and J_1 induce the same orientation on W^6.

Example 5. The structural equations for P_6 are given by

$$[X_1, X_2] = -\tfrac{1}{2}Z_1, \quad [X_1, X_4] = -\tfrac{1}{2}Z_2, \quad [X_2, X_3] = -\tfrac{1}{2}Z_2,$$

and correspond to a degeneration of (9). An abelian complex structure J on P_6 is defined by

$$JX_1 = X_2, \quad JX_3 = -X_4, \quad JZ_1 = -Z_2. \tag{11}$$

The associated Hermitian manifold $\Gamma\backslash P_6$ was studied in [1, §5], where it was shown that J is only one of a *finite* number of complex structures compatible with a fixed Riemannian metric.

3 Cohomology theory

In order to perform deformation theory on the compact complex nilmanifold X, we need to calculate cohomology with coefficients in the holomorphic tangent sheaf. We achieve this by identifying Dolbeault and Lie algebra cohomology, in the spirit of [13].

3.1 Lie algebra cohomology

With respect to a complex structure J, the complexified Lie algebra has a type decomposition. We may write

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}, \quad \mathfrak{t}_{\mathbb{C}} = \mathfrak{t}^{1,0} \oplus \mathfrak{t}^{0,1}, \quad \mathfrak{c}_{\mathbb{C}} = \mathfrak{c}^{1,0} \oplus \mathfrak{c}^{0,1}.$$

These are all spaces of left-invariant vectors on G. The definitions are extended to invariant (p,q)-forms in the standard way. For instance, $\bigwedge^k \mathfrak{g}_{\mathbb{C}}^{*(0,1)} = \mathfrak{g}^{*(0,k)}$ is the space of G-invariant $(0,k)$-forms.

Motivated by the property of the Chern connection on holomorphic tangent bundles [6], we define a linear operator $\overline{\partial}$ on $(0,1)$-vectors as follows. For any $(1,0)$-vector V and $(0,1)$-vector $\bar{U}$, set

$$\overline{\partial}_{\bar{U}} V := [\bar{U}, V]^{1,0}.$$

We obtain a linear map

$$\overline{\partial} : \mathfrak{g}^{1,0} \to \mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0}.$$

In view of (3),

$$\overline{\partial}_{\overline{T}_k} T_j = [\overline{T}_k, T_j]^{1,0} = \sum_{\alpha} E^{\alpha}_{kj} W_\alpha, \tag{12}$$

whence

$$\overline{\partial} T_j = \sum_{k,\alpha} E^{\alpha}_{kj} \overline{\omega}^k \otimes W_\alpha \quad \text{and} \quad \overline{\partial} W_\alpha = 0.$$

Extend this definition to a linear map on $\mathfrak{g}^{*(0,k)} \otimes \mathfrak{g}^{1,0}$ by setting

$$\overline{\partial}(\overline{\omega} \otimes V) = \overline{\partial}\overline{\omega} \otimes V + (-1)^k \overline{\omega} \wedge \overline{\partial} V,$$

where $V \in \mathfrak{g}^{*(0,k)}$ and $V \in \mathfrak{g}^{1,0}$. For instance, any element $\boldsymbol{\mu}$ in $\mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0}$ can be written as

$$\boldsymbol{\mu} = \sum_{i,j} \mu^i_j \overline{\omega}^j \otimes T_i + \sum_{i,\alpha} \mu^i_\alpha \overline{\omega}^\alpha \otimes T_i + \sum_{j,\beta} \mu^\beta_j \overline{\omega}^j \otimes W_\beta + \sum_{\alpha,\beta} \mu^\beta_\alpha \overline{\omega}^\alpha \otimes W_\beta. \tag{13}$$

By Lemma 1 and (12),

$$\begin{aligned} -\overline{\partial}\boldsymbol{\mu} &= \sum_{i,j} \mu^i_j \overline{\omega}^j \wedge \overline{\partial} T_i + \sum_{i,\alpha} \mu^i_\alpha \overline{\omega}^\alpha \wedge \overline{\partial} T_i \\ &= \sum_{i,j,k,\beta} \mu^i_j E^\beta_{ki} \overline{\omega}^j \wedge \overline{\omega}^k \otimes W_\beta + \sum_{i,k,\alpha,\beta} \mu^i_\alpha E^\beta_{ki} \overline{\omega}^\alpha \wedge \overline{\omega}^k \otimes W_\beta. \end{aligned}$$

This calculation gives us a necessary and sufficient condition for $\boldsymbol{\mu}$ to be $\overline{\partial}$-closed, which we now record as

Lemma 2. *Suppose that an element $\boldsymbol{\mu}$ in $\mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0}$ is given by formula (13). Then $\overline{\partial}\boldsymbol{\mu} = 0$ if and only if*

$$\sum_i (\mu^i_j E^\alpha_{ki} - \mu^i_k E^\alpha_{ji}) = 0 \quad \textit{and} \quad \sum_i \mu^i_\alpha E^\beta_{ji} = 0,$$

for each j, k, α, β.

We have a sequence

$$0 \to \mathfrak{g}^{1,0} \to \mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0} \to \ \cdots \ \to \mathfrak{g}^{*(0,k-1)} \otimes \mathfrak{g}^{1,0} \stackrel{\overline{\partial}_{k-1}}{\to} \mathfrak{g}^{*(0,k)} \otimes \mathfrak{g}^{1,0} \stackrel{\overline{\partial}_k}{\to} \cdots$$

The next result comes as no surprise, reflecting as it does the fact that our $\overline{\partial}$ operators are the natural ones induced on invariant differential forms.

Lemma 3. *The above sequence is a complex, i.e. $\overline{\partial}_k \circ \overline{\partial}_{k-1} = 0$ for all $k \geqslant 1$.*

Proof: It suffices to verify the lemma for $k = 1$. Let $\{\overline{\omega}^p\}, \{œ_q\}$ be dual bases of $\mathfrak{g}^{*(1,0)}$ and $\mathfrak{g}^{1,0}$, where the indices p, q run over the entire range $1, \ldots, n+m$. By definition,

$$\overline{\partial} V = \sum_p \overline{\omega}^p \otimes [œ_p, V]^{1,0}.$$

Applying $\overline{\partial}$ again,

$$\overline{\partial}^2 V = \sum_p \overline{\partial}\overline{\omega}^p \otimes [œ_p, V]^{1,0} - \sum_{p,q} (\overline{\omega}^p \wedge \overline{\omega}^q) \otimes [œ_q, [œ_p, V]^{1,0}]^{1,0}.$$

Since $[\oe_p, \oe_q]^{1,0} = 0$, we can delete the penultimate projection ^{1,0} above. The Jacobi identity

$$[\oe_p, [\oe_q, V]] - [\oe_q, [\oe_p, V]] = [[\oe_p, \oe_q], V]$$

implies that

$$\sum_{p,q} \overline{\omega}^p \wedge \overline{\omega}^q \otimes [\oe_q, [\oe_p, V]]^{1,0} = -\tfrac{1}{2}\sum_{p,q} \overline{\omega}^p \wedge \overline{\omega}^q \otimes [[\oe_p, \oe_q], V]^{1,0}.$$

If σ is a $(1,0)$-form, we can now contract σ with $\overline{\partial}^2 V$ to obtain the following form of type $(0,2)$:

$$\begin{aligned} \sigma(\overline{\partial}^2 V) &= \sum_p \sigma([\oe_p, V])\overline{\partial}\overline{\omega}^p + \tfrac{1}{2}\sum_{p,q} \sigma([[\oe_p, \oe_q], V])(\overline{\omega}^p \wedge \overline{\omega}^q) \\ &= -2\sum_p d\sigma(\oe_p, V)\overline{\partial}\overline{\omega}^p - \sum_{p,q} d\sigma([\oe_p, \oe_q], V)(\overline{\omega}^p \wedge \overline{\omega}^q). \end{aligned}$$

For this to vanish for all V and σ, we need to show that

$$2\sum_{r=1}^{n+m} \overline{\partial}\overline{\omega}^r \otimes \oe_r = -\sum_{p,q} (\overline{\omega}^p \wedge \overline{\omega}^q) \otimes [\oe_p, \oe_q].$$

This equation amounts to stating that the $\overline{\omega}^p \wedge \overline{\omega}^q$ component of $2\overline{\partial}\oe^r$ equals

$$\overline{\omega}^r([\oe_p, \oe_q]) = 2d\overline{\omega}^r(\oe_p, \oe_q) = 2\overline{\partial}\overline{\omega}^r(\oe_p, \oe_q),$$

which is correct. QED

Definition 1. *Define $H^k_{\overline{\partial}}(\mathfrak{g}^{1,0})$ to be the kth cohomology* $\ker \overline{\partial}_k / \operatorname{Im} \overline{\partial}_{k-1}$ *of the above complex; more precisely,*

$$H^k_{\overline{\partial}}(\mathfrak{g}^{1,0}) = \frac{\ker\left(\overline{\partial}_k : \mathfrak{g}^{*(0,k)} \otimes \mathfrak{g}^{1,0} \to \mathfrak{g}^{*(0,k+1)} \otimes \mathfrak{g}^{1,0}\right)}{\overline{\partial}_{k-1}\left(\mathfrak{g}^{*(0,k-1)} \otimes \mathfrak{g}^{1,0}\right)}.$$

We shall interpret these spaces geometrically in the next subsection.

3.2 Dolbeault cohomology

Let Γ be a J-invariant co-compact lattice in G, and $X = \Gamma\backslash G$ the associated nilmanifold parameterizing left cosets. Let $\psi\colon G \to G/C$ be the quotient map, where C is the center of G. Since G is 2-step abelian, G/C is abelian. In terms of the abelian varieties $F := C/C \cap \Gamma$ and $M := \psi(G)/\psi(\Gamma)$, we obtain a holomorphic fibration

$$\Psi\colon X \longrightarrow M$$

with fiber F.

Lemma 4. *Let $\mathcal{O}_X$ and Θ_X be the structure sheaf and the tangent sheaf of X. For $p \geqslant 1$, the direct image sheaves with respect to Ψ are*

$$R^p\Psi_*\mathcal{O}_X = \bigwedge^p \mathfrak{c}^{*(0,1)} \otimes \mathcal{O}_M = \mathfrak{c}^{*(0,p)} \otimes \mathcal{O}_M,$$
$$R^p\Psi_*\Psi^*\Theta_M = \bigwedge^p \mathfrak{c}^{*(0,1)} \otimes \Theta_M = \mathfrak{c}^{*(0,p)} \otimes \Theta_M.$$

Proof: The second identity is a consequence of the first, and the projection formula. To prove the first, note that for any point m in M,

$$(R^p\Psi_*\mathcal{O}_X)_m = H^p(\Psi^{-1}(m), \mathcal{O}_X) \cong H^p(C, \mathcal{O}_X).$$

This has constant rank and, by Grauert's Theorem, the direct image sheaf is locally free. As $\Psi^{-1}(m)$ is isomorphic to a complex torus, for all $p \geqslant 1$,

$$H^p(\Psi^{-1}(m), \mathcal{O}_X) = \bigwedge^p H^1(\Psi^{-1}(m), \mathcal{O}_X),$$

The vector bundle $R^p\Psi_*\mathcal{O}_X$ is isomorphic to $\bigwedge^p R^1\Psi_*\mathcal{O}_X$. Since the space of vertical (0,1)-forms is trivialized by the left-invariant (0,1)-forms given in Lemma 1, we have

$$R^p\Psi_*\mathcal{O}_X \cong \bigwedge^p R^1\Psi_*\mathcal{O}_X \cong \bigwedge^p \mathfrak{c}^{*(0,1)} \otimes \mathcal{O}_M,$$

as required. QED

Lemma 5. *Let $\mathcal{O}_X$ and Θ_X be the structure sheaf and the tangent sheaf of X. Then*

$$H^k(X, \mathcal{O}_X) = \bigwedge^k \mathfrak{g}^{*(0,1)} = \mathfrak{g}^{*(0,k)},$$
$$H^k(X, \Psi^*\Theta_M) = \bigwedge^k (\mathfrak{g}^{*(0,1)}) \otimes \mathfrak{t}^{1,0} = \mathfrak{g}^{*(0,k)} \otimes \mathfrak{t}^{1,0}.$$

Proof: Consider the Leray spectral sequence with respect to the $\overline{\partial}$-operator and the holomorphic projection Ψ. One has

$$E_2^{p,q} = H^p(M, R^q\Psi_*\mathcal{O}_X), \qquad E_\infty^{p,q} \Rightarrow H^{p+q}(X, \mathcal{O}_X).$$

From the previous lemma, when $q \geqslant 1$,

$$\begin{aligned} E_2^{p,q} &= H^p(M, \bigwedge^q \mathfrak{c}^{*(0,1)} \otimes \mathcal{O}_M) = \bigwedge^q \mathfrak{c}^{*(0,1)} \otimes H^p(M, \mathcal{O}_M) \\ &= \bigwedge^q \mathfrak{c}^{*(0,1)} \otimes H^p(M, \mathcal{O}_M) = \bigwedge^q \mathfrak{c}^{*(0,1)} \otimes \bigwedge^p \mathfrak{t}^{*(0,1)}. \end{aligned}$$

Note that every element in $E_2^{p,q}$ is a linear combination of the tensor products of vertical $(0,q)$-forms and $(0,p)$-forms lifted from the base. Since these forms are globally defined and the differential d_2 is generated by the $\overline{\partial}$-operator, we have $d_2 = 0$. It follows that the Leray spectral sequence degenerates at the E_2-level. Therefore,

$$H^k(X, \mathcal{O}_X) = \bigoplus_{p+q=k} E_2^{p,q} = \bigwedge^k (\mathfrak{c}^{*(0,1)} \oplus \mathfrak{t}^{*(0,1)}) = \bigwedge^k \mathfrak{g}^{*(0,1)}.$$

Next, the spectral sequence for $\Psi^*\Theta_M$ gives

$$E_2^{p,q} = H^p(M, R^q\Psi_*\Psi^*\Theta_M), \qquad E_\infty^{p,q} \Rightarrow H^{p+q}(X, \Psi^*\Theta_M).$$

Moreover, $E_2^{p,q}$ is equal to

$$H^p(M, \textstyle\bigwedge^q \mathfrak{c}^{*(0,1)} \otimes \Theta_M) = \textstyle\bigwedge^q \mathfrak{c}^{*(0,1)} \otimes H^p(M, \Theta_M) = \textstyle\bigwedge^q \mathfrak{c}^{*(0,1)} \otimes \textstyle\bigwedge^p \mathfrak{t}^{*(0,1)} \otimes \mathfrak{t}^{1,0}.$$

Elements in $\mathfrak{t}^{1,0}$ are holomorphic vector fields on M and hence globally defined sections of $\Psi^*\Theta_M$ on X. Elements in $\bigwedge^q \mathfrak{c}^{*(0,1)}$ are pulled back to globally defined $(0,q)$-forms on X. Crucially, elements in $\bigwedge^p \mathfrak{t}^{*(0,1)}$ are globally-defined holomorphic $(0,p)$-forms on X, and the operator d_2 is identically zero. Therefore, the spectral sequence degenerates at E_2. We have

$$H^k(X, \Psi^*\Theta_M) = \bigoplus_{p+q=k} E_2^{p,q} = \textstyle\bigwedge^k (\mathfrak{c}^{*(0,1)} \oplus \mathfrak{t}^{*(0,1)}) \otimes \mathfrak{t}^{1,0} = \textstyle\bigwedge^k (\mathfrak{g}^{*(0,1)}) \otimes \mathfrak{t}^{1,0},$$

as required. QED

Theorem 1. *Let X be a 2-step nilmanifold with an abelian complex structure. There is a natural isomorphism $H^k(X, \Theta_X) \cong H^k_{\bar\partial}(\mathfrak{g}^{1,0})$.*

Proof: On the manifold X, we have the exact sequence

$$0 \to \mathfrak{c}^{1,0} \otimes \mathcal{O}_X \to \Theta_X \to \Psi^*\Theta_M \to 0.$$

A piece of the corresponding long exact sequence is

$$\to \mathfrak{c}^{1,0} \otimes H^k(X, \mathcal{O}_X) \to H^k(X, \Theta_X) \to H^k(X, \Psi^*\Theta_M) \xrightarrow{\delta_k} \mathfrak{c}^{1,0} \otimes H^{k+1}(X, \mathcal{O}_X) \to$$

¿From the last section, the coboundary map is

$$\delta_k : \mathfrak{g}^{*(0,k)} \otimes \mathfrak{t}^{1,0} \to \mathfrak{g}^{*(0,k+1)} \otimes \mathfrak{c}^{1,0},$$

and so

$$H^k(X, \Theta_X) \cong \ker \delta_k \oplus \frac{\mathfrak{g}^{*(0,k)} \otimes \mathfrak{c}^{1,0}}{\delta_{k-1}(\mathfrak{g}^{*(0,k-1)} \otimes \mathfrak{t}^{1,0})}.$$

We calculate the coboundary maps by chasing the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & T^{*(0,k+1)} \otimes \mathfrak{c}^{1,0} & \to & T^{*(0,k+1)} \otimes \Theta_X & \to & T^{*(0,k+1)} \otimes \Psi^*\Theta_M & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & T^{*(0,k)} \otimes \mathfrak{c}^{1,0} & \to & T^{*(0,k)} \otimes \Theta_X & \to & T^{*(0,k)} \otimes \Psi^*\Theta_M & \to & 0
\end{array}$$

The vertical maps are $\overline{\partial}$'s for Dolbeault cohomology. More specifically, if ∇ is the Chern connection, if ω is a $(0,k)$-form and V a vector field of type $(1,0)$, then

$$\overline{\partial}(\omega \otimes V) = \overline{\partial}\omega \otimes V + (-1)^k \omega \wedge \overline{\partial}^{\nabla} V.$$

In the following computation, we let $\{e_p\}$ be a left-invariant basis for $\mathfrak{g}^{1,0}$ and $\{\omega^p\}$ the dual basis.

Let $\overline{\omega}$ be a $(0,k)$-form. Let V be an element in $\mathfrak{t}^{1,0}$, considered as a holomorphic vector field on M and a holomorphic section of $\Psi^*\Theta_M$. Let $\tilde{V}$ a smooth lifting of this section to be a section of Θ_X. Then

$$\delta_k(\overline{\omega} \otimes V) = \overline{\partial}\overline{\omega} \otimes \tilde{V} + (-1)^k \sum_p \overline{\omega} \wedge \overline{\omega}^p \otimes [\overline{e}_p, \tilde{V}]^{1,0}.$$

The element T_j in $\mathfrak{g}_{\mathbb{C}}$ could be considered as holomorphic vector field on M. It could also be considered as a smooth vector field on X. Considering the latter a lifting of the former and applying the above formula, we see that $\delta_k = \overline{\partial}_k$ on $\mathfrak{g}^{*(0,k)} \otimes \mathfrak{t}^{1,0}$. Now $\overline{\partial}_k(\mathfrak{g}^{*(0,k)} \otimes \mathfrak{c}^{1,0}) = 0$, and so

$$\delta_{k-1}(\mathfrak{g}^{*(0,k-1)} \otimes \mathfrak{t}^{1,0}) = \overline{\partial}_{k-1}(\mathfrak{g}^{*(0,k-1)} \otimes \mathfrak{g}^{1,0}).$$

Since the Lie algebra $\mathfrak{g}$ is 2-step nilpotent, $\operatorname{Im}\overline{\partial}_{k-1} \subseteq \mathfrak{g}^{*(0,k)} \otimes \mathfrak{c}^{1,0}$. Also, we have

$$\ker \delta_k = \ker \overline{\partial}_k \cap (\mathfrak{g}^{*(0,k)} \otimes \mathfrak{t}^{1,0}).$$

Therefore,

$$\begin{aligned} H^k(X, \Theta_X) &= \ker \overline{\partial}_k \cap (\mathfrak{g}^{*(0,k)} \otimes \mathfrak{t}^{1,0}) \oplus \frac{\mathfrak{g}^{*(0,k)} \otimes \mathfrak{c}^{1,0}}{\operatorname{Im}\overline{\partial}_{k-1}} \\ &= \frac{\ker \overline{\partial}_k \cap (\mathfrak{g}^{*(0,k)} \otimes \mathfrak{t}^{1,0} \oplus \mathfrak{g}^{*(0,k)} \otimes \mathfrak{c}^{1,0})}{\operatorname{Im}\overline{\partial}_{k-1}} = H^k_{\overline{\partial}}(\mathfrak{g}), \end{aligned} \tag{14}$$

as stated. QED

To summarize the results in this section, we shall say that a tensor on X is *invariant* if its pull-back to G by the quotient map is invariant by left-translation by G. Lemma 5 and Theorem 1 then allow us to formulate

Theorem 2. *The Dolbeault cohomology on X with coefficients in the structure and tangent sheaf can be computed using invariant forms and invariant vectors.*

Although the above proof relies on the 2-step property, one might expect that this result has a more generally validity, at least in the nilpotent context. For an independent approach to this problem, see [4].

4 Deformation theory

We shall shortly be in a position to apply the Kuranishi method to construct deformations. But first, we shall exhibit harmonic representatives in the Dolbeault cohomology groups.

4.1 Harmonic theory

Theorem 1 reduces the question to finite-dimensional vector spaces, and we may choose an invariant Hermitian structure on X of the type mentioned mentioned after Lemma 1. We use the resulting inner product on $\mathfrak{g}^{*(0,k)} \otimes \mathfrak{g}^{1,0}$ to define the orthogonal complement of $\operatorname{Im} \overline{\partial}_{k-1}$ in $\ker \overline{\partial}_k$. Denote this space by $\operatorname{Im}^{\perp} \overline{\partial}_{k-1}$.

Theorem 3. *The space* $\operatorname{Im}^{\perp} \overline{\partial}_{k-1}$ *is a space of harmonic representatives for the Dolbeault cohomology* $H^k(X, \Theta_X)$ *on the compact complex manifold* X.

Proof: It suffices to prove that an element

$$\sum_p \overline{\sigma}^p \otimes e_p \in \operatorname{Im}^{\perp} \overline{\partial}_{k-1} \subseteq \mathfrak{g}^{*(0,k)} \otimes \mathfrak{g}^{1,0} \tag{15}$$

is $\overline{\partial}^*$-closed on the manifold X.

Any section of the trivial bundle over X with fibre $\mathfrak{g}^{*(0,k-1)} \otimes \mathfrak{g}^{1,0}$ is a sum of elements of the type $f\overline{\eta} \otimes V$, where f is a smooth function, $\overline{\eta} \in \mathfrak{g}^{*(0,k-1)}$ and $V \in \mathfrak{g}^{(1,0)}$. By Lemma 1, $\overline{\sigma}^p$ and $\overline{\eta}$ are $\overline{\partial}$-closed. Using double angular brackets for the L^2 inner product and summing over repeated indices, we calculate

$$\begin{aligned} \langle\!\langle \overline{\partial}^*(\overline{\sigma}^p \otimes e_p),\ f\overline{\eta} \otimes V \rangle\!\rangle &= \langle\!\langle \overline{\sigma}^p, \overline{\partial}(f\overline{\eta}) \rangle\!\rangle \langle e_p, V \rangle + (-1)^{k-1} \langle\!\langle \overline{\sigma}^p \otimes e_p,\ f\overline{\eta} \wedge \overline{\partial} V \rangle\!\rangle \\ &= \langle\!\langle \overline{\partial}^* \overline{\sigma}^p, f\overline{\eta} \rangle\!\rangle \langle e_p, V \rangle + (-1)^{k-1} \langle\!\langle \overline{\sigma}^p \otimes e_p,\ f\overline{\eta} \wedge \overline{\partial} V \rangle\!\rangle. \end{aligned}$$

The basis $\{\omega^i, \omega^\alpha\}$ of Lemma 1 determines a complex volume form that we may use to identify $\overline{\partial}^*$ with $\pm * \overline{\partial} *$, where $*$ is the corresponding $SU(n+m)$ invariant antilinear mapping $\mathfrak{g}^{*(0,k)} \to \mathfrak{g}^{*(0,n+m-k)}$. It follows that $\overline{\partial}^* \overline{\sigma} = 0$. The remaining term

$$\langle\!\langle \overline{\sigma}^p \otimes e_p,\ f\overline{\eta} \wedge \overline{\partial} V \rangle\!\rangle = \int_X \overline{f} \langle \overline{\sigma}^p \otimes e_p, \overline{\eta} \wedge \overline{\partial} V \rangle = \langle \overline{\sigma}^p \otimes e_p, \overline{\partial}(\overline{\eta} \otimes V) \rangle \int_X \overline{f}$$

vanishes by assumption (15). QED

Corollary 1. *Let* $\boldsymbol{\mu} \in \mathfrak{g}^{*(0,k)} \otimes \mathfrak{g}^{1,0}$. *Then* $\overline{\partial}^* \boldsymbol{\mu}$ *with respect to the* L_2*-norm on the compact manifold* X *is equal to* $\overline{\partial}^* \boldsymbol{\mu}$ *with respect to the Hermitian inner product on the finite-dimensional vector spaces* $\mathfrak{g}^{*(0,k)} \otimes \mathfrak{g}^{1,0}$.

Proof: This follows from the displayed formulae in the previous proof. QED

4.2 The Schouten-Nijenhuis bracket

If $\overline{\omega} \otimes V$ and $\overline{\omega}' \otimes V'$ are vector-valued (0,1)-forms representing elements in $H^1(X, \Theta_X)$, their product with respect to the Schouten-Nijenhuis bracket is a vector-valued (0,2)-form

$$\{\cdot, \cdot\} : H^1(X, \Theta_X) \times H^1(X, \Theta_X) \to H^2(X, \Theta_X).$$

It is defined at the level of forms by

$$\{\overline{\omega} \otimes V, \overline{\omega}' \otimes V'\} = \overline{\omega}' \wedge L_{V'}\overline{\omega} \otimes V + \overline{\omega} \wedge L_V \overline{\omega}' \otimes V' + \overline{\omega} \wedge \overline{\omega}' \otimes [V, V'].$$

Via the isomorphism with Lie algebra cohomology, elements in $H^1(X, \Theta_X)$ lie in $\mathrm{Im}^\perp \overline{\partial}_0$. Since the vector and form parts are all left-invariant, $\iota_V \overline{\omega}'$ is a constant. Therefore, $L_V \overline{\omega}' = d\iota_V \overline{\omega}' + \iota_V d\overline{\omega}' = \iota_V d\overline{\omega}'$, and

$$\{\overline{\omega} \otimes V, \overline{\omega}' \otimes V'\} = \overline{\omega}' \wedge \iota_{V'} d\overline{\omega} \otimes V + \overline{\omega} \wedge \iota_V d\overline{\omega}' \otimes V' + \overline{\omega} \wedge \overline{\omega}' \otimes [V, V'].$$

The complex structure is abelian, so $[V, V'] = 0$ for all $(1,0)$-vectors, and

$$\{\overline{\omega} \otimes V, \overline{\omega}' \otimes V'\} = \overline{\omega}' \wedge \iota_{V'} d\overline{\omega} \otimes V + \overline{\omega} \wedge \iota_V d\overline{\omega}' \otimes V'. \tag{16}$$

Using the vector space direct sum $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{c}$, we write

$$\mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0} = (\mathfrak{t}^{*(0,1)} \otimes \mathfrak{t}^{1,0}) \oplus (\mathfrak{c}^{*(0,1)} \otimes \mathfrak{t}^{1,0}) \oplus (\mathfrak{c}^{*(0,1)} \otimes \mathfrak{c}^{1,0}) \oplus (\mathfrak{t}^{*(0,1)} \otimes \mathfrak{c}^{1,0}). \tag{17}$$

If $\overline{\omega} \otimes V \in \mathfrak{t}^{*(0,1)} \otimes \mathfrak{c}^{1,0}$ then $d\overline{\omega} = 0$, because all elements in $\mathfrak{t}^{*(k,l)}$ are closed. On the other hand, $d\overline{\omega}' \in \mathfrak{t}^{*(1,1)}$. Since $\iota_V d\overline{\omega}' = 0$ for $V \in \mathfrak{c}^{1,0}$, we have

$$\{\mathfrak{t}^{*(0,1)} \otimes \mathfrak{c}^{1,0},\ \mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0}\} = 0. \tag{18}$$

In order to compute $\{\boldsymbol{\mu}, \boldsymbol{\nu}\}$ on $\mathrm{Im}^\perp \overline{\partial}_0$, we compute the bracket amongst elements in the obvious basis. In view of (18), we need to calculate the brackets arising from the first three summands in (17). There are six types of bracket to calculate. Since $\overline{\omega}^k$ and $\overline{\omega}^j$ are closed,

$$\begin{gathered}
\{\overline{\omega}^j \otimes T_i, \overline{\omega}^k \otimes T_l\} = 0 \\
\{\overline{\omega}^j \otimes T_i, \overline{\omega}^\alpha \otimes T_l\} = \overline{\omega}^j \wedge \iota_{T_i} d\overline{\omega}^\alpha \otimes T_l = -\overline{E}^\alpha_{ik} \overline{\omega}^j \wedge \overline{\omega}^k \otimes T_l \\
\{\overline{\omega}^j \otimes T_i, \overline{\omega}^\alpha \otimes W_\sigma\} = \overline{\omega}^j \wedge \iota_{T_i} d\overline{\omega}^\alpha \otimes W_\sigma = -\overline{E}^\alpha_{ih} \overline{\omega}^j \wedge \overline{\omega}^h \otimes W_\sigma \\
\{\overline{\omega}^\alpha \otimes T_l, \overline{\omega}^\beta \otimes T_j\} = -\overline{E}^\beta_{lh} \overline{\omega}^\alpha \wedge \overline{\omega}^h \otimes T_j - \overline{E}^\alpha_{jh} \overline{\omega}^\beta \wedge \overline{\omega}^h \otimes T_l \\
\{\overline{\omega}^\alpha \otimes T_l, \overline{\omega}^\beta \otimes W_\gamma\} = \overline{\omega}^\alpha \wedge \iota_{T_l} d\overline{\omega}^\beta \otimes W_\gamma = -\overline{E}^\beta_{lh} \overline{\omega}^\alpha \wedge \overline{\omega}^h \otimes W_\gamma \\
\{\overline{\omega}^\alpha \otimes W_\beta, \overline{\omega}^\gamma \otimes W_\delta\} = 0.
\end{gathered} \tag{19}$$

The above formulae allow us to calculate $\{\boldsymbol{\mu},\boldsymbol{\nu}\}$. If $\boldsymbol{\mu}$ is given in coordinates as in (13) and $\boldsymbol{\nu}$ similarly then, suppressing summation signs, we have

$$\begin{aligned}
\{\boldsymbol{\mu},\boldsymbol{\nu}\} \;=\;& -(\mu^i_j\nu^\ell_\alpha+\nu^i_j\mu^\ell_\alpha)\overline{E}^\alpha_{ik}\,\overline{\omega}^j\wedge\overline{\omega}^k\otimes T_\ell\\
&-(\mu^\ell_\alpha\nu^j_\beta+\nu^\ell_\alpha\mu^j_\beta)(\overline{E}^\beta_{\ell k}\overline{\omega}^\alpha\wedge\overline{\omega}^k\otimes T_j+\overline{E}^\alpha_{jk}\overline{\omega}^\beta\wedge\overline{\omega}^k\otimes T_\ell)\\
&-(\mu^i_j\nu^\delta_\gamma+\nu^i_j\mu^\delta_\gamma)\overline{E}^\gamma_{ik}\,\overline{\omega}^j\wedge\overline{\omega}^k\otimes W_\delta \qquad (20)\\
&-(\mu^i_\alpha\nu^\delta_\gamma+\nu^i_\alpha\mu^\delta_\gamma)\overline{E}^\gamma_{ik}\,\overline{\omega}^\alpha\wedge\overline{\omega}^k\otimes W_\delta \qquad (21)
\end{aligned}$$

In particular,

$$\begin{aligned}
\{\boldsymbol{\mu},\boldsymbol{\mu}\} \;=\;& -2\mu^i_j\mu^\ell_\alpha\overline{E}^\alpha_{ik}\,\overline{\omega}^j\wedge\overline{\omega}^k\otimes T_\ell\\
&-2\mu^\ell_\alpha\mu^j_\beta(\overline{E}^\beta_{\ell k}\overline{\omega}^\alpha\wedge\overline{\omega}^k\otimes T_j+\overline{E}^\alpha_{jk}\overline{\omega}^\beta\wedge\overline{\omega}^k\otimes T_\ell) \qquad (22)\\
&-2\mu^i_j\mu^\delta_\gamma\overline{E}^\gamma_{ik}\,\overline{\omega}^j\wedge\overline{\omega}^k\otimes W_\delta-2\mu^i_\alpha\mu^\delta_\gamma\overline{E}^\gamma_{ik}\,\overline{\omega}^\alpha\wedge\overline{\omega}^k\otimes W_\delta,
\end{aligned}$$

which is of course is an element of $\mathfrak{g}^{*(0,2)}\otimes\mathfrak{g}^{1,0}$.

4.3 Kuranishi theory

To construct deformations, we apply Kuranishi's recursive formula. Let $\{\beta_1,\ldots,\beta_N\}$ be an orthonormal basis of the harmonic representatives of $H^1(X,\Theta_X)$. For any vector $\mathbf{t}=(t_1,\ldots,t_N)$ in $\mathbb{C}^N$, let

$$\boldsymbol{\mu}(\mathbf{t})=t_1\beta_1+\cdots+t_N\beta_N. \qquad (23)$$

We set $\boldsymbol{\phi}_1=\boldsymbol{\mu}$, and next define $\boldsymbol{\phi}_r$ inductively for $r\geqslant 2$.

Consider the $\overline{\partial}$-operator on X with respect to the Hermitian metric h previously defined, its adjoint operator $\overline{\partial}^*$, and the Laplacian

$$\triangle=\overline{\partial}\overline{\partial}^*+\overline{\partial}^*\overline{\partial}. \qquad (24)$$

Let $\mathcal{G}$ be the corresponding Green's operator that inverts $\triangle$ on the orthogonal complement of the space of harmonic forms, and let $\{\ ,\ \}$ denote the Schouten-Nijenhuis bracket. Then we set

$$\boldsymbol{\phi}_r(\mathbf{t})=\tfrac12\sum_{s=1}^{r-1}\overline{\partial}^*\mathcal{G}\{\boldsymbol{\phi}_s(\mathbf{t}),\boldsymbol{\phi}_{rs}(\mathbf{t})\}=\tfrac12\sum_{s=1}^{r-1}\mathcal{G}\overline{\partial}^*\{\boldsymbol{\phi}_s(\mathbf{t}),\boldsymbol{\phi}_{r-s}(\mathbf{t})\}, \qquad (25)$$

and consider the formal sum

$$\boldsymbol{\Phi}(\mathbf{t})=\sum_{r\geqslant 1}\boldsymbol{\phi}_r. \qquad (26)$$

Let $\{\gamma_1, \ldots, \gamma_M\}$ be an orthonormal basis for the space of harmonic $(0,2)$-forms with values in Θ_X. Define $f_k(\mathbf{t})$ to be the L^2-inner product $\langle\langle\{\mathbf{\Phi}(\mathbf{t}), \mathbf{\Phi}(\mathbf{t})\}, \gamma_k\rangle\rangle$. Kuranishi theory asserts the existence of $\epsilon > 0$ such that

$$\{\mathbf{t} \in \mathbb{C}^N : |\mathbf{t}| < \epsilon,\ f_1(\mathbf{t}) = 0, \ldots, f_M(\mathbf{t}) = 0\} \tag{27}$$

forms a locally complete family of deformations of X. We shall denote this set by Kur. For each $\mathbf{t} \in \mathrm{Kur}$, the associated sum $\mathbf{\Phi} = \mathbf{\Phi}(\mathbf{t})$ satisfies the integrability condition

$$\overline{\partial}\mathbf{\Phi} + \tfrac{1}{2}\{\mathbf{\Phi}, \mathbf{\Phi}\} = 0 \tag{28}$$

that now follows from (25) and the definition of $\mathcal{G}$.

More explicitly, we may treat $\mathbf{\Phi}$ is a linear map from $(0,1)$-vectors to $(1,0)$-vectors. It determines a complex structure on our manifold X whose distribution of (0,1)-vectors is given by

$$\begin{cases} \overline{S}_j = \overline{T}_j + \mathbf{\Phi}(\overline{T}_j), \\ \overline{V}_\alpha = \overline{W}_\alpha + \mathbf{\Phi}(\overline{W}_\alpha). \end{cases} \tag{29}$$

This set of equations is analogous to the gauge-theoretic defininition of a connection as $d_A = d + A$, where A is a matrix of 1-forms. In principal bundle language, d_A determines a horizontal distribution formed from the flat one by adding A as a vertical component. Then (28) is the analogue of setting the curvature of d_A to be zero, and assures us that the new distribution (29) is closed under Lie bracket.

We are now ready to make precise the first theorem of the Introduction:

Theorem 4. *Let G be a 2-step nilpotent Lie group with co-compact subgroup Γ, and let J be an abelian invariant complex structure on $X = \Gamma\backslash G$. Then the deformations arising from J parameterized by (27) are all invariant complex structures.*

Proof: It suffices to show that every term in the power series (26) lies in $\mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0}$. We shall prove this by induction. By Theorem 2, $\boldsymbol{\phi}_1 = \boldsymbol{\mu}$ belongs to this space.

Assume that $\boldsymbol{\phi}_s \in \mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0}$ for all $1 \leqslant s \leqslant r-1$. The computations of §4.2 show that $\{\boldsymbol{\phi}_s, \boldsymbol{\phi}_{r-s}\}$ is always contained in

$$\mathfrak{g}^{*(0,2)} \otimes \mathfrak{g}^{1,0} = \mathfrak{g}^{*(0,2)} \otimes \mathfrak{g}^{1,0} = \operatorname{Im}\overline{\partial}_1 \oplus \operatorname{Im}^\perp \overline{\partial}_1.$$

Let π_0 denote projection to the subspace $\operatorname{Im}\overline{\partial}_1$.

By Theorem 3, the component $\operatorname{Im}^\perp \overline{\partial}_1$ is the harmonic part of $H^2(X, \Theta)$. Since (24) satisfies $\triangle \circ \overline{\partial} = \overline{\partial} \circ \triangle$, and $\operatorname{Im}\overline{\partial}_1$ is orthogonal to the harmonic part, $\triangle$ maps $\operatorname{Im}\overline{\partial}_1$ isomorphically onto itself. It follows that

$$\mathcal{G}\{\boldsymbol{\phi}_s, \boldsymbol{\phi}_{r-s}\} = \mathcal{G}\pi_0\{\boldsymbol{\phi}_s, \boldsymbol{\phi}_{r-s}\} \subseteq \operatorname{Im}\overline{\partial}_1.$$

In particular, it is an invariant tensor. Corollary 1 shows that $\overline{\partial}^*\mathcal{G}\{\boldsymbol{\phi}_s, \boldsymbol{\phi}_{r-s}\}$ is again an invariant tensor. The same is true of $\boldsymbol{\phi}_r$. By induction, (26) is an infinite series of invariant tensors. QED

5 Deformations leading to abelian structures

In the light of Theorem 4, we are now ready to identify deformations of J leading not just to invariant complex structures, but to *abelian* ones.

Given an element $\boldsymbol{\mu} = \boldsymbol{\mu}(\mathbf{t})$ in the virtual parameter space $H^1(X, \Theta_X)$ as in (23), we apply the preceding method to generate the infinite series (26), and consider (29). For the latter to define an *abelian* complex structure, the Lie bracket of any pair of $(0,1)$-vectors must in fact vanish identically. In this case, we shall say that $\boldsymbol{\mu}$ generates an *abelian deformation*. Such an assumption leads to the following equations:

$$\begin{aligned} \big[\overline{S}_j, \overline{S}_k\big] &= 0, \qquad 1 \leqslant j, k \leqslant n & (30)\\ \big[\overline{S}_j, \overline{V}_\alpha\big] &= 0, \qquad 1 \leqslant j \leqslant n, \quad n+1 \leqslant \alpha \leqslant n+m & (31)\\ \big[\overline{V}_\alpha, \overline{V}_\beta\big] &= 0, \qquad n+1 \leqslant \alpha, \beta \leqslant n+m. & (32) \end{aligned}$$

Since $\overline{W}_\alpha$ is in the center,

$$[\overline{V}_\alpha, \overline{V}_\beta] = [\boldsymbol{\Phi}(\overline{W}_\alpha), \boldsymbol{\Phi}(\overline{W}_\beta)],$$

and this vanishes since the original complex structure is abelian. Therefore, equation (32) is satisfied automatically.

Let us examine the infinitesimal consequence of the first two equations. Let t represent a real variable, and replace $\boldsymbol{\mu}$ by $t\boldsymbol{\mu}$ so that $\boldsymbol{\Phi}$ becomes $\sum t^r \boldsymbol{\phi}_r$. Then in the notation of (13), equation (30) leads to

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \big[\overline{S}_j, \overline{S}_k\big] = \big[\overline{T}_j, \boldsymbol{\phi}_1 \overline{T}_k\big] + \big[\boldsymbol{\phi}_1 \overline{T}_j, \overline{T}_k\big] = \big[\overline{T}_j, \mu_k^i T_i\big] + \big[\mu_j^i T_i, \overline{T}_k\big] \\ &= (\mu_k^i E_{ji}^\alpha - \mu_j^i E_{ki}^\alpha) W_\alpha + (\mu_k^i F_{ji}^\alpha - \mu_j^i F_{ki}^\alpha) \overline{W}_\alpha. \qquad (33) \end{aligned}$$

The coefficient of W_α vanishes when $\overline{\partial}\boldsymbol{\mu} = 0$, by Lemma 2. Equation (31) leads to

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \big[\overline{S}_j, \overline{V}_\alpha\big] = \big[\overline{T}_j, \boldsymbol{\phi}_1 \overline{W}_\alpha\big] = \big[\overline{T}_j, \mu_\alpha^i T_i\big] \\ &= \mu_\alpha^i E_{ji}^\beta W_\beta + \mu_\alpha^i F_{ji}^\beta \overline{W}_\beta. \qquad (34) \end{aligned}$$

The coefficient of W_β is again 0 when $\boldsymbol{\mu}$ is $\overline{\partial}$-closed.

The above calculations give a set of necessary conditions limiting the type of deformations that one needs to consider. They motivate

Definition 2. *A form $\boldsymbol{\mu}$ given in coordinates as in (13) satisfies* Condition A *if*

$$\sum_i (\mu_j^i F_{ki}^\alpha - \mu_k^i F_{ji}^\alpha) = 0 \quad \textit{and} \quad \sum_i \mu_\alpha^i F_{ji}^\beta = 0,$$

for each j, k, α, β.

It is striking that these conditions are completely analogous to those of Lemma 2. In view of (33) and (34), we can now state

Proposition 1. *A parameter $\boldsymbol{\mu}$ represents an infinitesimal abelian deformation if and only if it is $\overline{\partial}$-closed and satisfies Condition A.*

Next suppose that $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are vector-valued 1-forms that are both $\overline{\partial}$-closed and satisfy Condition A. Since $\overline{E}^{\alpha}_{ij} = -F^{\alpha}_{ji}$, every term in (21) is equal to 0. For example, the first term $-\mu^i_j \nu^\ell_\alpha \overline{E}^{\alpha}_{ik} \overline{\omega}^j \wedge \overline{\omega}^k \otimes T_\ell$ is equal to

$$-\mu^i_j \nu^\ell_\alpha F^{\alpha}_{ki} \overline{\omega}^j \wedge \overline{\omega}^k \otimes T_\ell = -\nu^\ell_\alpha \left(\mu^i_j F^{\alpha}_{ki} - \mu^i_k F^{\alpha}_{ji}\right) \overline{\omega}^j \otimes \overline{\omega}^k \otimes T_\ell = 0.$$

and similarly every term in $\{\boldsymbol{\mu}, \boldsymbol{\nu}\}$ is equal to zero. In particular, $\{\boldsymbol{\mu}, \boldsymbol{\mu}\} = 0$.

Using the recursive formula (25), the higher order terms are all equal to zero, and so the series $\boldsymbol{\Phi}$ and $\boldsymbol{\mu}$ coincide by construction. Furthermore, $\{\boldsymbol{\Phi}, \boldsymbol{\Phi}\} = \{\boldsymbol{\mu}, \boldsymbol{\mu}\} = 0$, and there is no additional obstruction to integrability. Therefore,

Proposition 2. *On a 2-step nilmanifold X with abelian complex structure, an element in $H^1(X, \Theta_X)$ is infinitesimally abelian only if it is integrable to a 1-parameter family of abelian complex structures.*

Our main result concerning the deformation of abelian complex structures is

Theorem 5. *On a 2-step nilmanifold with abelian complex structure, a parameter $\boldsymbol{\mu}$ in $\mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0}$ generates an abelian deformation if and only if it is $\overline{\partial}$-closed and satisfies Condition A.*

Proof: If $\boldsymbol{\mu}$ generates an abelian deformation, it is infinitesimally abelian. By Proposition 1, the form is $\overline{\partial}$-closed and satisfies Condition A.

Conversely, if $\boldsymbol{\Phi}$ is $\overline{\partial}$-closed, it represents a cohomology class in $H^1(X, \Theta)$. Since it also satisfies Condition A, it is infinitesimally abelian. By Proposition 2, it represents an integrable abelian complex structure. QED

5.1 Fully abelian deformations

We are curious to know when the entire virtual parameter space $H^1(X, \Theta_X)$ integrates to abelian complex structures.

Theorem 6. *Let $X = \Gamma\backslash G$ be a compact 2-step nilmanifold endowed with an abelian complex structure. Suppose that every direction of the virtual parameter space is integrable to a 1-parameter family of abelian complex structures and that the dimension of the center of Lie algebra $\mathfrak{g}$ is equal to 1. Then $\mathfrak{g}$ is isomorphic to the direct sum of a Heisenberg algebra and a 1-dimensional abelian algebra.*

Proof: Given the hypothesis on the center, we may as well drop the index α in E^{α}_{ij}. This feature makes the subsequent construction possible.

Given the structural constants, for each set of j, k, l, m, choose an element $\boldsymbol{\mu}$ in $\mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0}$ by setting $\mu^l_k = E_{km}$ and $\mu^m_j = E_{jl}$ and all other terms are set to zero. By Lemma 2, each such $\boldsymbol{\mu}$ is closed. By equation (23), such $\boldsymbol{\mu}$ satisfies the equation $\{\boldsymbol{\mu}, \boldsymbol{\mu}\} = 0$ and therefore there is no obstruction for it to represent an integrable complex structure.

By hypothesis, $\boldsymbol{\mu}$ represents an abelian complex structure. By Condition A,

$$E_{km}\overline{E}_{lj} - E_{jl}\overline{E}_{mk} = 0.$$

It follows that $|E_{km}|^2 = |E_{mk}|^2$ for all k and m.

If every E_{km} vanishes then the algebra is abelian. On the other hand, if at least one E_{km} is non-zero, then $E_{mk} \neq 0$. For every $E_{jl} \neq 0$, the ratio

$$\frac{\overline{E}_{jl}}{E_{lj}} = \frac{\overline{E}_{km}}{E_{mk}}$$

is independent of the choice of j, l. Hence, there exists a real number θ such that

$$e^{i\theta} E_{jl} = \overline{E}_{lj}, \tag{35}$$

for every pair of indices (j, l). It follows that

$$\left[\overline{T}_j, T_l\right] = E_{jl}W + F_{jl}\overline{W} = E_{jl}W - \overline{E}_{lj}\overline{W} = E_{jl}W - e^{i\theta}E_{jl}\overline{W}.$$

Choosing

$$D_{jl} = e^{i(\pi+\theta)/2}E_{jl}, \quad U = e^{-i(\pi+\theta)/2}W$$

gives

$$\left[\overline{T}_j, T_l\right] = D_{jl}(U + \overline{U}).$$

With (35), we find that the matrix (D_{jl}) is skew-Hermitian. If we now choose a basis of $(0,1)$-vectors so that the matrix D is diagonal, the diagonal entries are purely imaginary or zero. The restriction on the central dimension forces the matrix D to be a constant multiple of the identity matrix. It follows that the structural equations exactly mirror those of the Heisenberg algebra as seen in (6). QED

Example 6. There exist examples satisfying the first hypothesis of the theorem, but not the second. To see this, take $\mathfrak{g}$ to be the real 8-dimensional Lie algebra with non-zero complex structural equations

$$\left[\overline{T}_1, T_1\right] = W_3 + \overline{W}_3, \quad \left[\overline{T}_2, T_2\right] = W_4 + \overline{W}_4,$$

and real 4-dimensional center. By Lemma 2, $\mu_1^2 = \mu_2^1 = 0$, and $\mu_\alpha^h = 0$ for $1 \leqslant h \leqslant 2$ and $3 \leqslant \alpha \leqslant 4$. It follows that

$$H^1(X, \Theta_X) = \langle \overline{\omega}^1 \otimes T_1,\, \overline{\omega}^2 \otimes T_2,\, \overline{\omega}^1 \otimes W_4,\, \overline{\omega}^2 \otimes W_3 \rangle,$$

and one may check that each direction is integrable to abelian complex structures. Globally, the associated compact complex manifold is the product of two primary Kodaira surfaces.

6 Six-dimensional structures

In dimension 6, there are precisely six classes of 2-step groups or nilmanifolds with an abelian complex structure [14]. Namely, the abelian group R^6, the product $H_5 \times R^1$ of a 5-dimensional Heisenberg group with a 1-dimensional group, the product $H_3 \times R^3$ of the 3-dimensional Heisenberg group with a 3-dimensional abelian group, the product $H_3 \times H_3$ of two 3-dimensional Heisenberg groups, and the groups P_6 and W_6. These were encountered in §2.1.

We shall use Example 4 to illustrate that results in this article produce adequate information for finding the parameters for integrable and abelian deformations. Using (10), consider the basis

$$T_1 = X_1 - iX_2, \quad T_2 = X_3 + iX_4, \quad W = Z_5 + iZ_6$$

of $\mathfrak{g}^{1,0}$; the corresponding basis for $\mathfrak{g}^{*(0,1)}$ is $\{\overline{\omega}^1, \overline{\omega}^2, \overline{\omega}\}$. The structural equations yield

$$[\overline{T}_1, T_2] = -W,$$

so that

$$E_{12} = -1, \quad F_{21} = 1,$$

and all other structural constants are equal to zero. In particular,

$$d\overline{\omega} = \omega^1 \wedge \overline{\omega}^2, \quad \iota_{T_1} d\overline{\omega} = \overline{\omega}^2, \quad \iota_{T_2} d\overline{\omega} = 0. \tag{36}$$

Mimicking the proof of Lemma 2, any element $\boldsymbol{\mu} \in \mathfrak{g}^{*(0,1)} \otimes \mathfrak{g}^{1,0}$ can be written as

$$\boldsymbol{\Phi} = \mu_j^i \overline{\omega}^j \otimes T_i + \mu_3^i \overline{\omega} \otimes T_i + \mu_j^3 \overline{\omega}^j \otimes W + \mu_3^3 \overline{\omega} \otimes W,$$

and

$$\overline{\partial} \boldsymbol{\mu} = \mu_2^2 \overline{\omega}^2 \wedge \overline{\omega}^1 \otimes W + \mu_3^2 \overline{\omega} \wedge \overline{\omega}^1 \otimes W. \tag{37}$$

This shows that $\boldsymbol{\mu}$ is closed if and only if $\mu_2^2 = \mu_3^2 = 0$. Since $\overline{\partial} T_2 = -\overline{\omega}^1 \otimes W$, the space of harmonic elements is in the orthogonal complement of $\overline{\omega}^1 \otimes W$. Therefore,

$$\dim H^1(X, \Theta_X) = \dim\{\mu_2^2 = \mu_3^2 = \mu_1^3 = 0\} = 6.$$

Using Definition 2, we see immediately that $\boldsymbol{\mu}$ satisfies Condition A if and only if

$$\mu_1^1 = \mu_3^1 = 0. \tag{38}$$

The number of parameters corresponding to abelian deformations is therefore 4.

Alternatively, we can count the number of integrable parameters, disregarding the abelian issue. To do so, we employ the recursive formula from §4.3, and first calculate the self-bracket of a harmonic representative $\boldsymbol{\mu}$. Using (36),(16), (19), we deduce that

$$\begin{aligned}\{\boldsymbol{\mu},\boldsymbol{\mu}\} &= \{\mu_1^1\overline{\omega}^1\otimes T_1+\mu_3^1\overline{\omega}\otimes T_1+\mu_3^3\overline{\omega}\otimes W,\ \mu_1^1\overline{\omega}^1\otimes T_1+\mu_3^1\overline{\omega}\otimes T_1+\mu_3^3\overline{\omega}\otimes W\}\\ &= \mu_3^1(2\mu_1^1\overline{\omega}^1\wedge\overline{\omega}^2\otimes T_1+\mu_3^1\overline{\omega}\wedge\overline{\omega}^2\otimes T_1+2\mu_3^3\overline{\omega}\wedge\overline{\omega}^2\otimes W)-2\mu_1^1\mu_3^3\overline{\partial}(\overline{\omega}^2\otimes T_2).\end{aligned}$$

Using (25), we take

$$\boldsymbol{\phi}_2 = \mu_1^1\mu_3^3\overline{\omega}^2\otimes T_2.$$

This quadratic correction term exactly corresponds to the equation $d=-av$ in [8, Proposition 4.2].

If we set $\boldsymbol{\Phi}=\boldsymbol{\mu}+\boldsymbol{\phi}_2$, then (28) becomes

$$\mu_3^1(2\mu_1^1\overline{\omega}^1\wedge\overline{\omega}^2\otimes T_1+\mu_3^1\overline{\omega}\wedge\overline{\omega}^2\otimes T_1+2\mu_3^3\overline{\omega}\wedge\overline{\omega}^2\otimes W)=0.$$

The resulting deformation is therefore integrable if and only if $\mu_3^1=0$, so there is a total of 5 integrable parameters. As predicted by Thereom 5, the obstruction μ_3^1 already features in the abelian equations (38).

Let $\mathfrak{g}$ denote a real 6-dimensional nilpotent Lie algebra admitting a complex structure. The table in [14, Appendix] displays, for each such $\mathfrak{g}$, the complex dimension of the space $\mathcal{C}(\mathfrak{g})$ of *invariant* complex structures at a smooth point of one of its connected component. This was done with little regard for when complex structures are equivalent, in the knowledge that subsequent work would clarify the findings. The following table compares these computations with results yielded by the techniques of this paper.

The last five columns display the complex dimension

(i) d of $\mathcal{C}(\mathfrak{g})$,
(ii) h^0 of the space $\dim H^0(X,\Theta_X)$ of infinitesimal automorphisms,
(iii) h^1 of the virtual parameter space $H^1(X,\Theta_X)$,
(iv) of the space Kur of (27), or the number of integrable parameters,
(v) of the subspace Abel of Kur describing abelian deformations.

relative to the complex structures defined by (5),(7),(8),(10),(11), for each of the last five rows in turn.

	d	h^0	h^1	dim Kur	dim Abel
T^6	9	3	9	9	9
$(\Gamma\backslash H_5)\times S^1$	6	1	4	4	4
$(\Gamma\backslash H_3)\times T^3$	7	2	6	6	6
$(\Gamma\backslash H_3)\times(\Gamma\backslash H_3)$	6	1	4	4	3
$\Gamma\backslash W_6$	6	2	6	5	4
$\Gamma\backslash P_6$	6	1	4	4	3

Table

If J is an invariant complex structure with unobstructed deformations on a nilmanifold, $\mathcal{C}(\mathfrak{g})$ has the same dimension as the kernel of

$$\overline{\partial}:\mathfrak{g}^{*(0,1)}\otimes\mathfrak{g}^{1,0}\to\mathfrak{g}^{*(0,2)}\otimes\mathfrak{g}^{1,0},$$

whereas $\dim\mathrm{Kur}=h^1$. Since the dimension of the image $\overline{\partial}(\mathfrak{g}^{*(0,0)}\otimes\mathfrak{g}^{1,0})$ equals $3-h^0$, we deduce further that

$$d=3-h^0+h^1$$

if J is a generic point of $\mathcal{C}(\mathfrak{g})$.

At points of $\mathcal{C}(\mathfrak{g})$ where h^0 jumps to a higher value, the Kuranishi method is unable to detect the additional equivalences that come into play at neighbouring points where the symmetry group drops. Consequently, we can only assert that $d+h^0-3$ is an *upper bound* for $\dim\mathrm{Kur}$. In the Table, these two numbers only disagree for W_6, and this is because J_1 was 'too' special a choice at which to carry out the computations. If we work instead at a nearby point J' corresponding to $\mu_3^3\neq 0$, then $h^0=1$ and the dimensions of Kur and Abel drop to 4 and 3. This is because the orbit $J'\cdot W_6$ under right translation by the group has dimension 2, whereas $\dim(J\cdot W_6)=1$.

A more extreme example, not tabulated, is that of the non-abelian complex structure J_0 on $\Gamma\backslash W_6$ for which $h^0=3$, $h^1=d=\dim\mathrm{Kur}=6$ and $\dim\mathrm{Abel}=0$ [12, 14].

6.1 Final examples

In general, information on H^1 and abelian deformations can be extracted algebraically using Lemmas 2 and Lemma 2. The computation of Kur is more challenging, though it is useful to realize that every parameter is integrable when $h^1=\dim\mathrm{Abel}$.

In our last two examples, the first applies the theory of §4.3, whereas the second replies on this theory to pass directly to a calculation with invariant differential forms.

Example 7. For $(\Gamma\backslash H_3)\times(\Gamma\backslash H_3)$ (see Example 3), we may take

$$T_1=\tfrac12(X_1-iY_1),\quad T_2=\tfrac12(X_2-iY_2),\quad W=\tfrac12(Z_1-iZ_2).$$

The associated complex structural equations are

$$[\overline{T}_1,T_1]=-\tfrac12 i(W+\overline{W}),\qquad [\overline{T}_2,T_2]=\tfrac12(W-\overline{W}).$$

In terms of the dual basis $\{\omega^i\}$, any harmonic representative of H^1 is a linear combination of

$$\overline{\omega}^1\otimes T_1,\quad \overline{\omega}^2\otimes T_2,\quad \overline{\omega}\otimes W,\quad \overline{\omega}^1\otimes T_2+i\overline{\omega}^2\otimes T_1.$$

If

$$\boldsymbol{\mu}=\mu_1^1\overline{\omega}^1\otimes T_1+\mu_2^2\overline{\omega}^2\otimes T_2+\mu_3^3\overline{\omega}\otimes W+\mu_1^2(\overline{\omega}^1\otimes T_2+i\overline{\omega}^2\otimes T_1),$$

we obtain

$$\boldsymbol{\Phi}=\boldsymbol{\mu}-\mu_3^3\mu_1^2(\overline{\omega}^1\otimes T_2-i\overline{\omega}^2\otimes T_1).$$

Then (29) defines an integrable complex structure.

Example 8. The complex structural equations corresponding to Example 5 can be written in the form $d\omega^1=0=d\omega^2$ and

$$2d\omega^3=i\omega^1\wedge\overline{\omega}^1+\omega^1\wedge\overline{\omega}^2-\overline{\omega}^1\wedge\omega^2=i\omega^{1\overline{1}}+\omega^{1\overline{2}}-\omega^{\overline{1}2},$$

in which the last expression is an abbreviation of the middle one. By [8, Theorem 1.1], any invariant complex structure J' sufficiently near to J has a basis of $(1,0)$ forms that can be written

$$\left\{\begin{array}{l}\alpha^1=\omega^1+\Phi_1^1\overline{\omega}^1+\Phi_2^1\overline{\omega}^2\\ \alpha^2=\omega^2+\Phi_1^2\overline{\omega}^1+\Phi_2^2\overline{\omega}^2\\ \alpha^3=\omega^3+\Phi_1^3\overline{\omega}^1+\Phi_2^3\overline{\omega}^2+\Phi_3^3\overline{\omega}^3.\end{array}\right. \tag{39}$$

This is a dual version of (29), and the integrabilty condition (28) amounts to the assertion that $(d\alpha^3)^{0,2}=0$, or equivalently

$$\begin{array}{rcl}0&=&2d\alpha^3\wedge\alpha^1\wedge\alpha^2\\ &=&[(i\omega^{1\overline{1}}+\omega^{1\overline{2}}-\omega^{\overline{1}2})+\Phi_3^3(i\omega^{1\overline{1}}-\omega^{1\overline{2}}+\omega^{\overline{1}2})]\wedge[-\Phi_2^1\omega^{2\overline{2}}+\Phi_2^2\omega^{1\overline{2}}+\Phi_1^1\omega^{\overline{1}2}]\\ &=&[-i\Phi_2^1(1+\Phi_3^3)+(1-\Phi_3^3)(\Phi_1^1-\Phi_2^2)]\omega^{1\overline{1}2\overline{2}}.\end{array}$$

Thus, (39) defines an integrable complex structure on condition that

$$i(1+\Phi_3^3)\Phi_2^1=(1-\Phi_3^3)(\Phi_1^1-\Phi_2^2),$$

and the coefficients in (39) are sufficiently small (in particular, $|\Phi_3^3|<1$).

In this case, the Kuranishi series (26) is infinite, as it is not possible to express one coefficient as a polynomial in the others. The term $\Phi^3_1\overline{\omega}^1 + \Phi^3_2\overline{\omega}^2$ can be reduced to zero by a suitable right translation of J, and therefore plays no role in the equivalence problem. It follows that $\dim \mathrm{Kur} = 4$. The abelian condition

$$d\alpha^3 \wedge \overline{\alpha}^1 \wedge \overline{\alpha}^2 = 0$$

can be worked out in the same way, and forces $\Phi^1_2 = 0$ and $\Phi^1_1 = \Phi^2_2$, so $\dim \mathrm{Abel} = 3$.

The Table and examples allow us to infer that:

(i) It is possible that every direction in the virtual parameter space is integrable but only some are tangent to abelian deformation. This occurs for $(\Gamma\backslash H_3)\times(\Gamma\backslash H_3)$ and $\Gamma\backslash P_6$.

(ii) It is also possible that some directions are obstructed, irrespective of the abelian condition. An example is $\Gamma\backslash W_6$. This phenomenon was described in [14, Lemma 4.3], and contrasts with the unobstructed deformation theory for $(\Gamma\backslash W_6, J_0)$.

(iii) The centers of T^6 and $(\Gamma\backslash H_3)\times T^3$ certainly have dimension greater than 1, and these examples do not therefore contradict Theorem 6.

All these observations demonstrate the subtle dependence of $\dim \mathrm{Kur}$ and $\dim \mathrm{Abel}$ on the underlying algebraic structure of the group G.

The techniques of this paper can in theory be applied to study deformations of the compact quotients of the six series of 2-step nilmanifolds with abelian complex structures in any complex dimension.

Other work of the authors shows that in many cases an explicit description of Kur, and indeed a global moduli space, is possible [7, 10]. It is also realistic to seek to describe the quotient of the space $\mathcal{C}(\mathfrak{g})$ by the group of $\mathrm{Aut}(\mathfrak{g})$ of Lie algebra automorphisms of $\mathfrak{g}$, at least near a generic point of $\mathcal{C}(\mathfrak{g})$. In this case, in the W_6 example, $\mathcal{C}(\mathfrak{g})/\mathrm{Aut}(\mathfrak{g})$ is locally isomorphic to the quotient of Kur by the group of outer automorphisms of $\mathfrak{g}$ [5, §5].

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